Step 5. Moving the line of integration.

Now we can deduce

Theorem 6.33

$$\int_1^x \psi(t)\,dt = \frac{1}{x}x^2 + O\left(x^2\mathcal{E}(x)\right), \tag{40}$$

where $\mathcal{E}(x) = \exp\left(-c\log^{1/10} x\right)$ *for some constant* $c > 0$.

Look back in the Problem Sheets where is was shown that

$$x^{-\delta} \le \exp\left(-c\log^{1/10} x\right) \le (\log x)^{-A},$$

for any $\delta > 0$ and $A > 0$. Think of δ as small and A as large, so this says that $\mathcal{E}(x)$ tend to 0 slower than $x^{-\delta}$ however small δ might be, but tends to 0 quicker than $(\log x)^{-A}$, however large A might be.

Proof Recall the fundamental

$$\int_1^x \psi(t)\,dt = \frac{1}{2}x^2 - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\,\frac{x^{s+1}ds}{s\,(s+1)} + O\left(x\right),$$

for $c > 1$. With $T \ge 2$ to be chosen, truncate the integral at $\pm T$ and estimate the tail ends that are discarded by

$$\left|\int_{c+iT}^{c+i\infty} F(s)\,\frac{x^{s+1}ds}{s\,(s+1)}\right| \le \int_T^\infty |F(c+it)|\,\frac{x^{c+1}dt}{|c+it|\,|c+1+it|}.$$

Both $|c+it|$ and $|c+1+it| \ge |t|$ while $|F(c+it)| \ll \log^9 t$, so

$$\begin{aligned}\int_T^\infty |F(c+it)|\,\frac{x^{c+1}dt}{|c+it|\,|c+1+it|} &\ll x^{c+1}\int_T^\infty \frac{\log^9 t}{t^2}dt \\ &\ll \frac{x^{1+c}\log^9 T}{T}.\end{aligned} \tag{41}$$

(Recall the 'trick' explored in a problem sheet of estimating such integrals by splitting at T^2 and estimating each part separately.) In (41) choose $c = 1 + 1/\log x$ when

$$x^{1+c} = x^{2+1/\log x} = x^2 e^{\log x/\log x} = ex^2$$

and the error (41) is thus $\ll x^2 T^{-1}\log^9 T$. This leaves us with the integral along the vertical straight line from $c - iT$ to $c + iT$.

Next let $\mathcal{C}$ be the contour around the rectangle with corners at $c-iT$, $c+iT$, $1-\delta(T)+iT$ and $1-\delta(T)-iT$, where $\delta(T) = A/\log^9 T$. Here A is a constant chosen sufficiently small so that

$$\frac{A}{\log^9 T} \leq \frac{1}{2^{19}\left(\log T + 6\right)^9}.$$

Then $\zeta(s)$ will have no zeros and thus $F(s)$ no poles within or on this contour. So, by Cauchy's Theorem,

$$\frac{1}{2\pi i}\int_{\mathcal{C}} F(s)\,\frac{x^{s+1}ds}{s\,(s+1)} = 0.$$

That is

$$\frac{1}{2\pi i}\left(\int_{c-iT}^{c+it} + \int_{c+iT}^{1-\delta(T)+iT} + \int_{1-\delta(T)+iT}^{1-\delta(T)-iT} + \int_{1-\delta(T)-iT}^{c-iT}\right) F(s)\,\frac{x^{s+1}ds}{s\,(s+1)} = 0.$$

In both integrals over the **horizontal paths**, from $c+iT$ to $1-\delta(T)+iT$ and from $1-\delta(T)-iT$ to $c-iT$, we have $|s\,(s+1)| \geq T^2$. Therefore these integrals are bounded by

$$\ll \frac{\left(\log T\right)^9}{T^2}\int_{1-\delta(T)}^{c} x^{1+\sigma}d\sigma \ll \left(c-1+\delta(T)\right)\frac{x^{1+c}\log^9 T}{T^2},$$

simply bounding the integral by *length* $\times$ *largest value*. This contribution is dominated by (41).

Finally we have an integral on the **vertical line** from $1-\delta(T)-iT$ to $1-\delta(T)+iT$.

Let J_1 be the integral of $F(s)\,x^{s+1}/s\,(s+1)$ over $|t| \leq 2$ and J_2 the integral over $2 \leq |t| \leq T$. Then in the first integral $F(s)$ is bounded, by M, say, so

$$|J_1| \leq \int_{-2}^{2} Mx^{2-\delta(T)}\frac{dt}{\left(t+1\right)^2} \ll x^{2-\delta(T)}.$$

While, from Corollary 6.30,

$$J_2 \ll \int_{2}^{T} \left(\log t\right)^9 x^{2-\delta(T)}\frac{dt}{t^2} \ll x^{2-\delta(T)},$$

since the integral over t converges. Combine all these bounds as

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\,\frac{x^{s+1}ds}{s\,(s+1)} \ll \frac{x^2\log^9 T}{T} + x^{2-\delta(T)}. \tag{42}$$

Choose T to equalise (or balance) these two terms up to logarithmic factors (i.e. first forget about the $\log^9 T$ factor), which requires $T \approx x^{\delta(T)}$. Taking logarithms,

$$\log T \approx \frac{A \log x}{(\log T)^9}.$$

i.e. $T = \exp\left(c \log^{1/10} x\right)$ for some c. Then

$$x^{2-\delta(T)} = \frac{x^2}{T} = x^2 \exp\left(-c \log^{1/10} x\right). \tag{43}$$

The other error term in (42) has the $\log^9 T$ factor. Yet

$$\log^9 T = \left(c \log^{1/10} x\right)^9 \ll \exp\left(\varepsilon \log^{1/10} x\right)$$

for any $\varepsilon > 0$ (just take logarithms of both sides to see this). Then

$$\frac{x^2 \log^9 T}{T} \leq x^2 \exp\left(-(c - \varepsilon) \log^{1/10} x\right),$$

which is of the same form as (43) but with a slightly smaller constant c.

Collecting together we conclude that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^{s+1} ds}{s\,(s+1)} \ll x^2 \exp\left(-c \log^{1/10} x\right),$$

for some constant c. ■